Algebra PoW Packet
Symbol Logic

Problem 5212 • https://www.nctm.org/pows/

Welcome!

This packet contains a copy of the problem, the "answer check," our solutions, teaching suggestions, and some samples of the student work we received in November 2008. The text of the problem is included below. A print-friendly version is available using the "Print" link on the problem page.

Standards

In **Symbol Logic** the **key concepts** are number sense, systems with more variables than equations, and logical reasoning.

If your state has adopted the Common Core State Standards, this alignment may be helpful:

Grade 6: Expressions and Equations

5. Understand solving an equation or inequality as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.

Algebra: Creating Equations

1. Create equations and inequalities in one variable and use them to solve problems.
3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context.

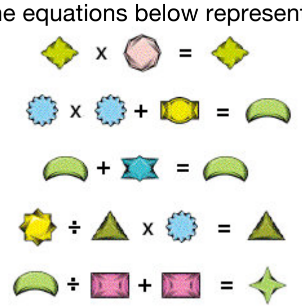
Algebra: Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
5. Use appropriate tools strategically.
7. Look for and make use of structure.

The Problem

Symbol Logic

The ten different symbols shown in the equations below represent the ten digits from 0 - 9:



Determine what digit each symbol represents. You might find it helpful to consider what each symbol can't possibly be, might possibly be, or definitely is. For your short answer, state the value of the diamond symbol in the first equation.

Be sure to fully and clearly explain the thinking and work you did to determine the value of each symbol.

Extra: Modular arithmetic reduces any number to the remainder when that number is divided by the "modulus". For example, the number 7 in mod 4 reduces to 3 because 3 is the remainder when 7 is divided by 4. Similarly, 10 mod 4 reduces to 2 since 2 is the remainder when 10 is divided by 4.

In mod 4 the only possible digits are 0, 1, 2, and 3 since those are the only possible remainders when you divide by 4.

Suppose the letters **A**, **B**, **C**, and **D** represent the four possible digits in mod 4. If **B + B = A** and **D + C = A**, what letter is **B + D** equal to? Why?

Answer Check

After students submit their solution, they can choose to "check" their work by looking at the answer that we provide. Along with the answer itself (which never explains how to actually **get** the answer) we provide hints and tips for those whose answer doesn't agree with ours, as well as for those whose answer does. You might use these as prompts in the classroom to help students who are stuck and also to encourage those who are correct to improve their explanation:

> The green diamond has a value of 7. Be sure you've found the values of the other symbols and clearly explained your thinking and work.

If your answer **doesn't match ours**:

- did you try using variables to represent the symbols and equations to make it easier to organize your work?
- did you look for which equations are easiest to start with?
- did you try starting out with what you know must be true?
- did you realize that you might be able to determine one symbol in an equation without getting the others until later?
- did you remember that each symbol represents a whole number from 0 to 9 and think about what that means when dividing two symbols?
- did you check your arithmetic?

If any of those ideas help you, you might *revise* your answer, and then leave a *comment* that tells us what you did. If you're still stuck, leave a *comment* that tells us where you think you need help or where you're having trouble.

If your answer **does match ours**,

- did you show and explain the thinking and work you did?
- is your explanation clear and complete? Would another student understand your solution?
- did you make any mistakes along the way? If so, how did you find and fix them?
- are there any hints that you would give another student?
- have you tried the Extra question?

Revise your work if you have any ideas to add. Otherwise leave us a *comment* that tells us how you think you did—you might answer one or more of the questions above.

Our Solutions

Symbol Logic offers students an opportunity to work with a set of equations in which there are fewer equations than variables and use logical reasoning to determine which values must be assigned to which variables.

Method 1: Logical Reasoning

The first thing I did was represent each symbol with a variable so it would be easier to work with. I just went through the equations and created a variable for each new symbol I encountered. Here's the list. Let:

```
a = the green diamond shape in the first equation
b = the pink octagon shape in the first equation
c = the blue asterisk shape in the second equation
d = the yellow band-aid shape in the second equation
e = the green moon shape in the second equation
f = the blue shape in the third equation
g = the yellow eight-pointed shape in the fourth equation
h = the green triangle shape in the fourth equation
i = the pink rectangle shape in the fifth equation
j = the green four-pointed shape in the fifth equation
```

Next I rewrote the five equations using my variables. I numbered the equations so I can refer to them more easily:

```
1)  a * b = a
2)  c * c + d = e
```

```
3)  e + f = e
4)  g ÷ h * c = h
5)  e ÷ i + i = j
```

The first thing I noticed was that in equation 3, adding f to e still makes e. That told me that f must equal 0.

Then I looked at equation 1. Since a times b is still a, there are two options. Either b is 1 since anything times 1 is itself, or a is 0 since 0 times anything is 0. But since I knew that f is 0 from equation 3, b must be 1.

With 0 and 1 accounted for, I moved to equation 2. It says that $c^2 + d = e$. Since they are all single digits, e can't be bigger than 9. With adding d to it, c^2 must be less than 9. Only 0, 1, and 2 have squares that are less than 9, and with 0 and 1 taken, c must be 2.

Now that c is 2, I substituted that into equation 2 and got $2^2 + d = e$, or d + 4 = e. With 0, 1, and 2 accounted for, the smallest d can be is 3, and the largest it can be is 5 since adding 4 to more than 5 won't give a single digit answer. So d is either 3, 4, or 5, and adding 4 to those, e is either 7, 8, or 9.

I also substituted 2 for c in equation 4, and got 2g/h = h. I multiplied both sides by h and had $2g = h^2$. Since 2 times whatever g is must be even, h must be an even number so that it will be even when squared. The smallest even number left is 4, so I tried 4, 6, and 8 for h:

```
2g = (4)^2        2g = (6)^2        2g = (8)^2
2g = 16           2g = 36           2g = 64
 g = 8             g = 18            g = 32
```

4 worked well with g also being a single-digit number, but 6 or 8 for h makes g too big. So h must be 4 and g must be 8.

Earlier I said that e must be 7, 8, or 9. Now that 8 is taken by g, e must be 7 or 9. Looking at equation 5, e is divided by i. Since that has to divide cleanly to make a single-digit result, i must be a factor of e. If e is 7, i would have to be 1 since that's the only factor of 7 besides 7 itself. But 1 is already taken, so e can't be 7. That leaves e being 9, and i being 3, the only factor of 9 still remaining. If e is 9 and i is 3, I can use equation 5 to calculate j:

```
e ÷ i + i = j
9 ÷ 3 + 3 = j
  3   + 3 = j
        6 = j
```

I said earlier that d + 4 = e, so if e is 9, d is 4 less than that or 5. So e must be 9, i must be 3, j must be 6, and d must be 5.

There is only one symbol left, the diamond from the first equation, or my variable a. There is also only one number left, 7, so a must be 7.

So my final result is:

```
a = 7 = the green diamond
b = 1 = the pink octagon
c = 2 = the blue asterisk
d = 5 = the yellow band-aid
e = 9 = the green moon
f = 0 = the blue shape in the third equation
g = 8 = the yellow eight-pointed shape
h = 4 = the green triangle
i = 3 = the pink rectangle
j = 6 = the green four-pointed shape
```

Checking all the equations by substituting those values:

```
1)  a * b = a        7 * 1 = 7            Yes
2)  c * c + d = e    2 * 2 + 5 = 9        Yes
3)  e + f = e        9 + 0 = 9            Yes
4)  g ÷ h * c = h    (8 ÷ 4) * 2 = 4      Yes
5)  e ÷ i + i = j    (9 ÷ 3) + 3 = 6      Yes
```

My answers check in all the equations.

Extra

I started with B + B = A and added all four possible digits for the B + B part:

0 + 0 = 0
1 + 1 = 2
2 + 2 = 0 (it makes 4 but that reduces to 0 mod 4)
3 + 3 = 2 (it makes 6 but that reduces to 2)

Since the answer (A) is a different digit than the one being added (B), the first one can't be right. But the other three all work, so I have three possibilities so far:

B = 1 and A = 2
B = 2 and A = 0
B = 3 and A = 2

Turning to the second equation, D + C = A, I tried each of the three cases from above:

First case: If B = 1 and A = 2, then C and D are 0 and 3 (in some order) and D + C = 0 + 3 = 3. Since A is 2, D + C does not equal A, and this case fails.

Second case: If B = 2 and A = 0, then C and D are 1 and 3 (in some order) and D + C = 1 + 3 = 0, which does equal A. This case is possible.

Third case: If B = 3 and A = 2, then C and D are 0 and 1 (in some order) and D + C = 0 + 1 = 1. Since A is 2, D + C does not equal A, and this case fails.

Now I know that only the second case is possible, so A = 0 and B = 2. I don't know which of C and D is the 1 and 3, so I can't evaluate B + D yet. I'll try both cases:

Case 1: A = 0, B = 2, C = 1, D = 3, and B + D = 2 + 3 = 1, which is C.

Case 2: A = 0, B = 2, C = 3, D = 1, and B + D = 2 + 1 = 3, which is C.

Even though I can't be sure whether C is 1 and D is 3 or vice-versa, in both cases the sum of B and D turns out to be the digit that is represented by C. The problem asked what letter B + D is equal to, so clearly B + D always equals C.

Method 2: Make a Grid and Logical Reasoning

My goal is to find the value for each symbol. I think I can use a grid to help me stay organized and figure out which symbols match which values 0 – 9. I decided to represent each symbol with a letter:

A = green diamond
B = pink circle
C = blue circle
D = yellow rectangle/oval
E = green moon
F = blue trapezoid
G = yellow star
H = dark green triangle
I = pink rectangle
J = light green star

I can rewrite the symbol equations as follows:

```
(1) A * B = A
(2) C * C + D = E
(3) E + F = E
(4) G / H * C = H
(5) E / I + I = J
```

Now that I have my variables clearly identified, I can draw a grid with symbols A-J and values 0-9:

	0	1	2	3	4	5	6	7	8	9
A										
B										
C										
D										
E										
F										
G										
H										
I										
J										

Then I can start trying to identify the symbol values by using clues from the five equations in the problem. I found my first clue in the third equation, which states: E + F = E. This means that F = E - E so F = 0 (regardless of the value of E).

Since I have found the value of the symbol represented as F, I can put it on my grid and cross out all other possible values for F and all other possible variables for 0:

	0	1	2	3	4	5	6	7	8	9
A	X									
B	X									
C	X									
D	X									
E	X									
F	0	X	X	X	X	X	X	X	X	X
G	X									
H	X									
I	X									
J	X									

The second clue I found was in the first equation, which states: A * B = A. Knowing that A cannot be 0 (since F is 0), I can divide by A to get B = A/A, so B would have to be 1. I can fill in more of my grid:

	0	1	2	3	4	5	6	7	8	9
A	X	X								
B	X	1	X	X	X	X	X	X	X	X
C	X	X								
D	X	X								
E	X	X								
F	0	X	X	X	X	X	X	X	X	X
G	X	X								
H	X	X								
I	X	X								
J	X	X								

Moving on to the second equation, I see that C * C + D = E. From this equation, I had a hunch that E was probably not going to be a small number because of squaring the C to start. Looking at my grid, I see that the smallest possible values for C and D are 2 and 3. If C is 3 then 3 * 3 is 9 and since D has to be at least 2 E will be at least 3 * 3 + 2 or 11, which is not a one-digit number. Since C can't be 3 or anything larger and have E work, C must be 2. That means 2 * 2 + D = E, so then E = D + 4. Now that C is 2, the smallest D can be is 3. I have to add 4 to D to get E and stay within the one-digit numbers, so D can be 3, 4, or 5, which makes E be 7, 8, or 9. I can update my chart to show all this new information:

	0	1	2	3	4	5	6	7	8	9
A	X	X	X							
B	X	1	X	X	X	X	X	X	X	X
C	X	X	2	X	X	X	X	X	X	X
D	X	X	X				X	X	X	X
E	X	X	X	X	X	X	X			
F	0	X	X	X	X	X	X	X	X	X
G	X	X	X							
H	X	X	X							
I	X	X	X							
J	X	X	X							

Having found many clues in the second equation of the main problem, I decided to look at equation 4. I am skipping equation 3, since there are no clues there to help me find the value of E.

Equation 4 states that G/H * C = H. I know C is 2, so that makes it G/H * 2 = H. Then I divided by 2 to get G/H = H/2. Since H has to be a whole number, H/2 will either be a whole number (when H is even) or end in .5 (when H is odd). That means G/H must also either be a whole number or end in .5 . According to my chart, both G and H can be any number from 3 to 9, so I went through all the possibilities and found every ratio of G/H that makes a whole number or a number that ends in .5 . I found the following choices:

$$G/H = 3/6,\ 4/8,\ 6/3,\ 6/4,\ 8/4,\ 9/3,\ 9/6$$

Then I checked each one to see if that result equaled the H value divided by 2 since G/H = H/2:

$$3/6 \neq 6/2,\ 4/8 \neq 8/2,\ 6/3 \neq 3/2,\ 6/4 \neq 4/2,\ 8/4 = 4/2,\ 9/3 \neq 3/2,\ 9/6 \neq 6/2$$

The only one that worked was when G = 8 and H = 4. To check my work I simply substitute those numbers in the fourth equation, which would look like this: 8/4 * 2 = 4. Simplifying, 2 * 2 = 4 and 4 = 4 so it checks. Now I can fill in more of my grid to show those values for G and H:

	0	1	2	3	4	5	6	7	8	9
A	X	X	X		X				X	
B	X	1	X	X	X	X	X	X	X	X
C	X	X	2	X	X	X	X	X	X	X
D	X	X	X		X		X	X	X	X
E	X	X	X	X	X	X	X		X	
F	0	X	X	X	X	X	X	X	X	X
G	X	X	X	X	X	X	X	X	8	X
H	X	X	X	X	4	X	X	X	X	X
I	X	X	X		X				X	
J	X	X	X		X				X	

Now I can move on to equation 5. Here it states that E / I + I = J. First, I see that the value for E must be 9 because looking at my grid I see that E can either be 7 or 9, and 7 cannot be divided by any possible value for I and come out as a whole number. Second, I see that if E is 9, then I must be 3, since it is the only possible number for I that 9 is divisible by. Having found the values of E and I, I can plug their values into equation 5 to find the value of J:

$$9/3 + 3 = J$$

3 + 3 = J
J = 6

Now I can fill up my grid knowing that E is 9, I is 3, and J is 6:

	0	1	2	3	4	5	6	7	8	9
A	X	X	X	X	X		X		X	X
B	X	1	X	X	X	X	X	X	X	X
C	X	X	2	X	X	X	X	X	X	X
D	X	X	X	X	X		X	X	X	X
E	X	X	X	X	X	X	X	X	X	9
F	0	X	X	X	X	X	X	X	X	X
G	X	X	X	X	X	X	X	X	8	X
H	X	X	X	X	4	X	X	X	X	X
I	X	X	X	3	X	X	X	X	X	X
J	X	X	X	X	X	X	6	X	X	X

Looking at my grid I see that D must be 5; meaning A is 7:

	0	1	2	3	4	5	6	7	8	9
A	X	X	X	X	X	X	X	7	X	X
B	X	1	X	X	X	X	X	X	X	X
C	X	X	2	X	X	X	X	X	X	X
D	X	X	X	X	X	5	X	X	X	X
E	X	X	X	X	X	X	X	X	X	9
F	0	X	X	X	X	X	X	X	X	X
G	X	X	X	X	X	X	X	X	8	X
H	X	X	X	X	4	X	X	X	X	X
I	X	X	X	3	X	X	X	X	X	X
J	X	X	X	X	X	X	6	X	X	X

So I have now found the value of every symbol.
The value of the diamond symbol (represented as A) is 7.

Teaching Suggestions

In ***Symbol Logic***, students are asked to explore a problem with more variables than equations and apply various clues about what numbers those variables might represent. There are a variety of approaches to this problem, but they all tend to fall within logical reasoning.

One interesting thing about this problem is that although there are more variables than equations, there is still a unique solution. Students who are familiar with systems of equations may be aware of the idea that when there are more variables than equations, the system cannot be solved. In fact, some students may set out to solve this problem as a system, making substitutions from one equation to another. I would not discourage that approach and would instead let them go with it until they reach a point of realizing that it is not getting them anywhere, then ask them what else they might try. One important problem-solving strategy is recognizing when a method is not working and deciding on an alternate approach.

So what makes this system of equations solvable? Clearly it's the constraint that all symbols represent digits from 0 to 9. Without that constraint, each individual equation would have infinite possible answers. Asking kids who are familiar with the situation where variables exceed equations to think about why this problem works can give rise to good discussion about systems in general. It's also a nice opportunity to introduce kids to a branch of equations that are known as Diophantine.

While many Diophantine equations require very complicated math to solve, the examples given in that article are appropriate for algebra students.

A nice way to start this problem is by thinking about what must be true, what can't be true and what might be true. Suppose your students have condensed the problem using variables and writing equations as we did in our sample solutions above. They will have something like this:

```
1)  a * b = a
2)  c * c + d = e
3)  e + f = e
4)  g / h * c = h
5)  e / i + i = j
```

Now have them think about what must, might, and can't be true based on those equations. They might generate lists like these:

Must be true:

- f must be 0 because in equation (3) it doesn't change anything when you add it to the other number.
- b must be 1 since in equation (1) it does not change anything when you multiply by it.
- there are special qualities in some of the other numbers that will make them fit certain equations (odd, even, etc.).
- e is a multiple of i

Can't be true:

- a can't be 0 if f must be 0
- i can't be bigger than e because e/i in equation (5) must make a whole number.
- c can't be bigger than 3 because of equation (2).
- e can't be prime because it is a multiple of i

Might be true:

- equation (1) works if either a is 0 or b is 1.
- h could go evenly into g.
- e is a big number.
- some equations like equation (2) and equation (4) have fewer number combinations than others.

Any such lists would be a great starting point for tackling this problem and moving forward. From here you could break students into groups or assign small pairs to learn more about particular variables and then regroup to see what else has been figured out.

Student Solutions

Focus on Completeness

In the solutions below, I've provided the scores the students would have received in the **Completeness** category of our scoring rubric. The table below is an excerpt from the rubric for this problem showing the guidelines for scoring in Completeness:

Novice	Apprentice	Practitioner	Expert
Has written very little that tells or shows how they found their answer.	Submitted explanation without work or work without explanation. Leaves out enough details that another student couldn't follow or learn from the explanation.	Explains all of the important steps taken to solve the problem. Shows equations, formulas, and calculations used and explains the rationale behind them. Defines variable(s).	Adds in useful extensions and further explanation of some of the ideas involved The additions are helpful, not just "I'll say more to get more credit."

For each solution, I've included a comment about why I would score it as shown, as well as what I'd ask the student to work on when they revise their solution to help them move forward with solving the problem or improving their write-up of their work.

Harold
age 13

Completeness
Novice

I tried to do this problem for a long time with my teacher and we could not finish it. We worked on it for a 45 minute class period.

Could not do.

Harold has shared very little about what he did and tried during class. I'd ask him to tell me everything he can about the first and second equations.

Brett
age 11

Completeness
Novice

Diamond = 7 rock= 1 Half moon= 9 moon=0 cloud=2 candy=5 star=8 triangle=4 rectangle=3 4 point star=6

That is my explanations

Brett's listed values for each of the symbols, which is terrific, but he hasn't given us any clue how he got those values. I would ask Brett to tell me which values he knew first and why and then how he used those values to find the others.

Marissa
age 13

Completeness
Apprentice

The diamond represents 5.

I kept filling in numbers for the shapes and saw what numbers would and wouldn't work. Then I tried different combinations and got my answer.

Marissa's strategy is actually explained clearly, but it's hard to know what she actually did. What shapes did she replace with what numbers and how did she check what worked and what didn't? I'd ask Marissa to tell me more about how she knew 5 worked for the diamond and if she noticed any patterns about what worked and what didn't.

Elizabeth
age 14

Completeness
Apprentice

The value of the diamond symbol in the first equation is 7.

We used guess and check and process of elimination to figure out what each symbol was. We knew 2 of them were 1 and 0. Then we went through with the remaining numbers and figured out which ones fit with each symbol.

Like Marissa, Elizabeth has done a nice job of identifying and even naming her strategy, but she too leaves out details that would help us really make sense of and follow her thinking. I would ask Elizabeth to tell me more about how she knew which two were 1 and 0 and how she figured out the remaining numbers.

Zachary
age 14

Completeness
Practitioner

The green diamond symbol is 7. The pink circle is 1, the blue fluff ball is 2, the yellow symbol is 5, the green crescent is 9, the blue star is 0, the yellow star is 8, the green triangle is 4, the pink rectangle is 3, and the green star is 6.

First I knocked off the obvious ones which were the pink circle (1) and the blue star (0). Then I solved the 5th problem down in which I decided that the yellow star has to be a number divisible by 3,4,5,6,7,8,or 9, so my only options were 9,3 or 6,3 or 8,4. Since the green triangle is the answer to that problem, the green triangle couldn't be 3, because there was no number times 3 or 2 that would equal 3 except 1, which is already used. So I went with 8 for the yellow star and 4 for the green triangle. Next I had to figure out the blue fluff ball which was 2 since 2x2 is 4. So on the 2nd equation I had to find the yellow symbol and green crescent. For the green crescent I knew it had to be a number divisible by 3 because of problem 5, which leaves me with 6 or 9. Since my number choice is down to 3,5, or 7 for the yellow symbol I had to choose 5 because 4+5=9. So the yellow symbol became 5 and the green crescent became 9, which also solved the 34d problem. Lastly, I had to solve the 5th problem, with only the numbers 3,6, and 7 left. I decided that the pink rectangle had to be 3 because it was the only number left that could be divided into 9. That made the green star 6. Since the last number left was 7 I used that as the green triangle in problem 1.

Reflection: This problem looked difficult at first, but as I slowly started to work on it made more sense. I found this answer with mostly logic and a little bit of guessing and checking.

Zachary has done a nice job of explaining his thinking, his reasoning and showing all of his steps. With him, I'd ask him to explain what made the pink circle and the blue star obvious. I'd also suggest that he add paragraph breaks to make his solution more readable and easier to follow.

Helen
age 13

Completeness
Practitioner

The green diamond in the first equation, a, is equal to 7.

First, I assigned variables to all of the symbols. I rewrote the equations so the problem now looked like this:

a x b = a
c x c + d = e
e + f = e
g / h x c = h
e / i + i = j

Helen's explanation is also easy to follow and she nicely walks us through how she used each equation to help her reason about what the possible values of each symbol (or variable in this case) could be and couldn't be. I would ask her what her dad helped her see that made things less confusing.

First, I looked at the first equation to solve this problem. This equation was a x b = a. Either a was 0 or b was 1. Looking at the third equation, I saw it was e + f = e, so that meant two things: f was 0; which meant that that zero was already taken, so b in the first equation would have to equal 1. On the side I wrote out the integers 0-9 and crossed a number out whenever it was taken, or assigned to a particular symbol. At this point only 0 and 1 were crossed out. Then I looked at the second equation. It was in the format c x c + d = e, which, simplified, was c^2 + d = e. Since the greatest value e could be was a 9, and they were all whole numbers, and 0 and 1 were already taken, c could only be 2. That made things easier. The equation was now 4 + d = e. D could have been 3, 4 or 5, and the e could have been 7, 8 or 9. Skipping the third equation and fourth equations for now, I started on the fifth equation. It was e / i + i = j. Since all of the numbers had to be whole numbers and 2 was already taken, then e must be 9 and i must be 3. Therefore, j was 6. With only 4, 5, 7, and 8 left, g in the fourth equation must be 8 and the h had to be 4. D in the second equation therefore had to be 5. The only number left was 7, so the a in the first equation, which could have been anything, must be 7.

REFLECTION: I looked at this problem for a long time and was thoroughly stumped until my dad showed me the first equation and explained the possibilities. We did them together in exactly the same order as I explained in my long answer. The a in the first equation confused me a little, but when i got down to just 7 left I saw the logic that a must be 7. I liked this equation and thought it was pretty fun, and I enjoyed the logic twist that it was given. When I had figured it all out, I saw the common sense in it and I thought it was a pretty clever problem.

Scoring Rubric

A **problem-specific rubric,** to help in assessing student solutions, is available in the Teacher Support Materials on the Problem page when you are logged in as a teacher. As shown above, we consider each category separately when evaluating the students' work, thereby providing more focused information regarding the strengths and weaknesses in the work.

https://www.nctm.org/contact-us/