Pre-Algebra PoW Packet

Lillian's Lines

December 1, 2008 • https://www.nctm.org/pows/

Welcome! This packet contains a copy of the problem, the "answer check," our solutions, teaching suggestions, and a problem-specific scoring rubric. The problem is new, so we have no student work to share.

The Problem In ***Lillian's Lines***, students are asked to find how many square panes a path goes through for a window of a given size.

The text of the problem is included below. A print-friendly version is available from the "Print this Problem" link on the current PreAlgPoW problem page.

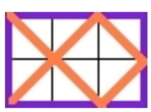
Lillian's Lines

I was bored one day, staring at a window, and I started imagining a continuous series of diagonals, going across the panes of the window. It reminded me of the path a pool ball would make, if you shot it from one corner at a 45° angle.

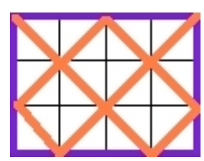
- I always start the path in the upper left corner, and travel at a 45° diagonal across the grid.
- If the path hits an edge of the outer rectangle, it bounces off at a 45° angle and continues its travel.
- The path continues in this way until it hits a corner of the outer rectangle (where a corner pocket would be on a pool table).

Here are the paths generated by some of the windows in my house:

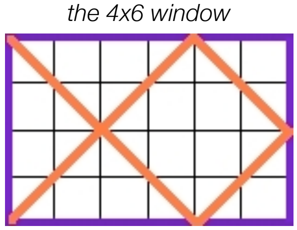
the 2x3 window



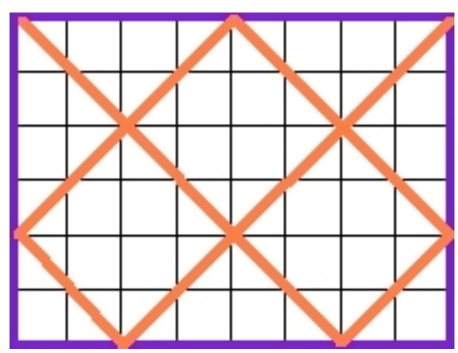
the 3x4 window



the 4x6 window



the 6x8 window



I noticed that sometimes the path goes through all of the little squares, and sometimes only some of them. I wondered how many of the square panes the path goes through for a window of a given size.

Question: How many square panes does the path go through in a 28 by 36 window?

Extra: What rule can you use for any size window?

Answer Check

The path would go through 252 panes.

If your answer does **not** match ours,

- did you notice any similarities between the paths of the 2x3 and 4x6 windows or the 3x4 and 6x8 windows?
- did you notice that sometimes the path goes through all the panes, and sometimes it doesn't?
- did you try some smaller windows, to make sure you understand the problem?
- did you notice any patterns?

If any of those ideas help you, you might *revise* your answer, and then leave a *comment* that tells us what you did. If you're still stuck, leave a *comment* that tells us where you think you need help.

If your answer **does** match ours,

- did you explain what made the problem hard?
- did you do anything to make the problem simpler to solve?
- is your explanation clear and complete?
- did you make any mistakes along the way? If so, how did you find them?
- what hints would you give another student trying to solve this problem?

Revise your work if you have any ideas to add. Otherwise leave us a *comment* that tells us how you think you did—you might answer one or more of the questions above.

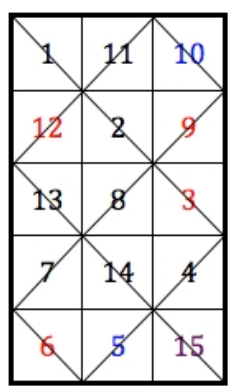
Our Solutions

The key concept of this problem is number theory. This is a great problem for noticing patterns and thinking about multiples, least common multiples, symmetry, and relatively prime numbers.

Method 1: Least Common Multiples

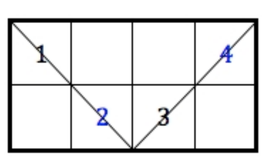
I decided to do like the girl in the story and try out some smaller windows to see if I could see any patterns before I tried the big window. I tried a 3 by 4 window, and as I drew the path, I counted panes aloud. Then I tried a 4 by 4 and a 5 by 4 window.

I noticed something when I was counting the panes aloud. The times when I hit a wall and changed direction were always multiples of one of the two dimensions. I tried a 5 by 3 to test my theory. Here is a drawing:



I labeled each pane with the number I said when I drew through it. I made the numbers red for when they hit the left or right walls, and blue for when they hit the top or bottom walls. Then the corner has to be purple.

Here's another example:



Do you see that the path always hits walls on multiples of the dimensions? Corners are hitting two walls at once, so the corner is always the first number that is a multiple of both dimensions (the least common multiple).

So, for the big window, I need to find the least common multiple of 36 and 28. I am going to list multiples of each, and see where they match.

28	36
56	72
84	108
112	144
140	180
168	216
196	252
224	288
252	

252 is the LCM of 28 and 36, so the path will go through 252 of the panes. That is not all of them, since 28 x 36 = 1008. So it doesn't even go through most of the panes.

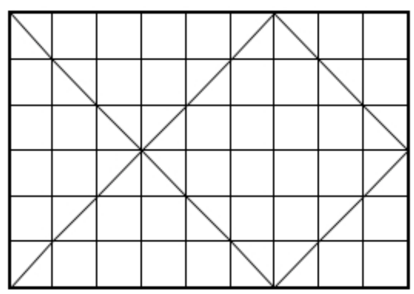
For the **Extra**, you just have to find the LCM of the two dimensions of the window, and that is the number of panes that the path goes through.

Method 2: Patterns and Comparing Shapes

I noticed from the pictures given in the problem that some windows have the same shape inside them, and some don't. The 2 by 3 and 4 by 6 windows both have fish shapes in them, and the 3 by 4 window has an upside-down pretzel shape in it. Before I tackled the big window, I wanted to use smaller windows to look for a pattern that I could apply to the big window.

I decided to see if I could find a pattern in what makes a fish shape in the window. I noticed 2 by 3 and 4 by 6 are multiples of each other. The 4 by 6 is twice as long and twice as tall as the 2 by 3.

I decided to draw a 6 by 9, predicting a fish shape.



Yup! And this fish has each part 3 times as long as the original 2 by 3 fish.

Let's see. . . the 2 by 3 fish went through 6 panes. The 4 by 6 fish went through 12 panes. And the 6 by 9 fish went through 18 panes. I'm starting to see a pattern. I can reduce both 4 by 6 and 6 by 9 to 2 by 3 – they are like equivalent fractions. The 2 by 3 rectangle can't be reduced at all, since 2 and 3 are relatively prime (so the fraction 2/3 is in "lowest terms). Then I keep track of by how much I reduced it. The 4 by 6 was reduced by a factor of 2, and the 6 x 9 was reduced by a factor of 3. To find the answer for 4 by 6, take the answer for 2 by 3 and multiply it by 2. For the 6 by 9, multiply it by 3.

Now, the big problem asks about a 28 by 36 grid. That's not a multiple of a 2 by 3 grid, since that would have to be 24 by 36. If I reduce the fraction 28/36, I get 7/9, having reduced by a factor of 4. So if I know the number of panes for a 7 by 9 rectangle, I just multiply it by 4 to get the answer.

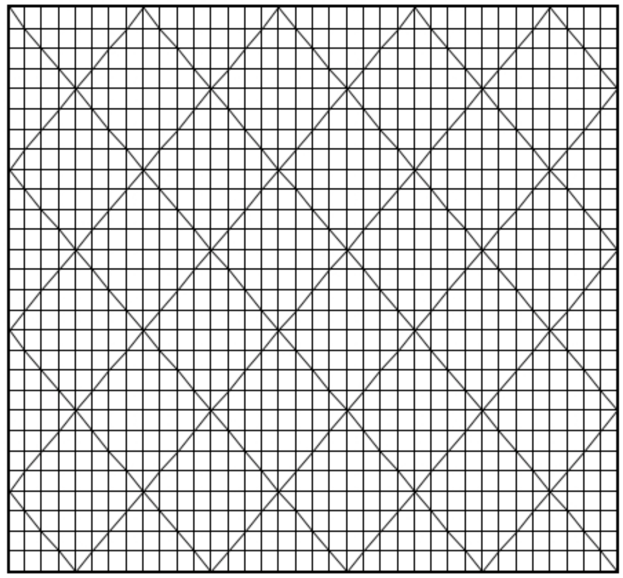
Let's see… 28 = 7 x 4. Four also goes into 36, 9 x 4. So I think the 28 by 36 grid will be four times as many panes as the 7 by 9 grid. I noticed earlier with the 2 by 3 and 3 by 4 windows that the paths went through all of the panes. The dimensions of those windows are relatively prime. I tried a 3 by 5 window, a 1 by 4 window, and a 2 by 5 window, and the path always went through all of the panes. So in the 7 by 9 window, the path is going to go through all the panes, and there are 63 panes.

Then, 28 by 36 is four times as big as 7 by 9, so the number of panes is four times as much.
4 x 63 = 252.

For the **Extra**, take the dimensions of the window and reduce them so that they are in "lowest terms," if they were a fraction. Multiply the two numbers, then multiply the result by the factor by which you reduced them. So for 70 by 100, 70/100 = 7/10, so that's 70 panes, and it was reduced by a factor of 10, so the answer is 700.

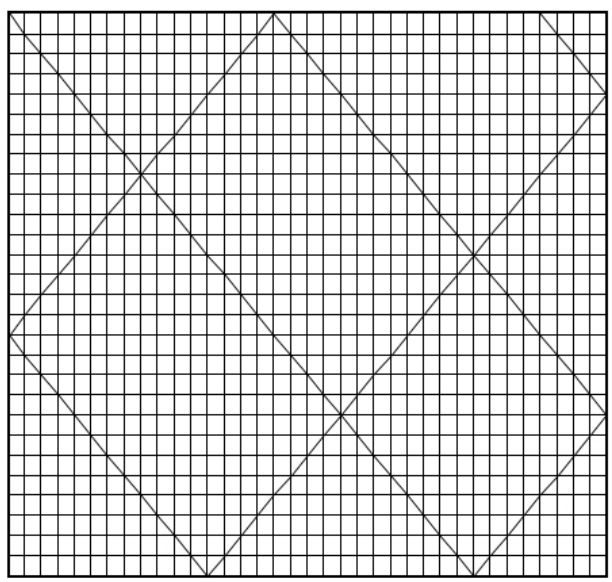
Method 3: Drawing Complete Picture and Counting

I think I understand the rules for drawing the path, so I can just draw out a 28 by 36 window.



I counted, and it goes through 252 panes.

Method 4: Noticing Pairs of Numbers with the Sum of 28

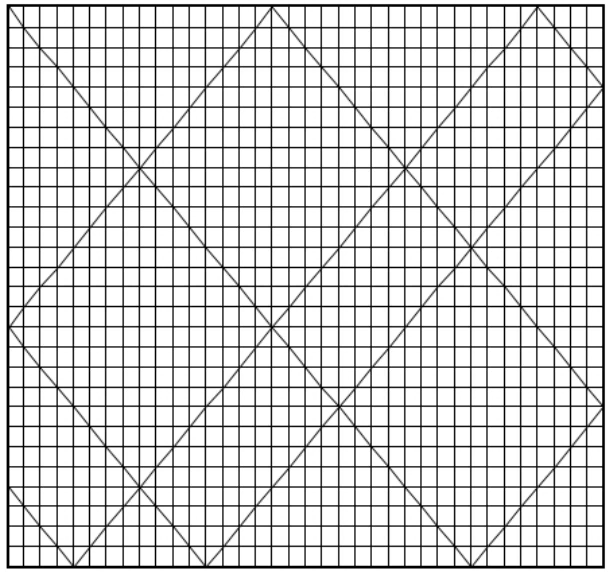


I started to draw out the 28 by 36 grid, but it was a lot of work. I decided to see if I could predict how many panes I would go through between turns.

I first went through 28, then hit a wall. That is the most panes possible, since I started at the top and ended up at the bottom. Then I went through 8 before hitting the next wall, and then 20, etc. The lengths I have so far are: 28, 8, 20, 16, 12, 24, and 4.

I noticed right away that after the 28, all the next pairs of numbers add up to 28 (8 + 20, 16 + 12, 24 + 4). I also noticed the first number in each pair is going up by 8.

This means that the next number should be 32, except that's longer than 28. So the next line must be 28.



Then I saw that the next line segment was 4 panes long, which is what I got before the last 28, so maybe now we're going backwards in the list – the next one will be 24 (since 4 + 24 = 28).

So far I have 28, 8, 20, 16, 12, 24, 4, 28, and next get 4, 24 (to make 28), 12 (8 more than 4), 16, 20, 8, 28. Then, since you're at 28, which was the beginning of the list, there's nothing you can add to make 28, so I think you're done.

The list is sort of symmetric – you make long skinny boxes that get wider and wider and shorter and shorter, then eventually they start to get longer and narrower again. The 28 in the middle is the "turning point."

That means the number of panes is

$$\mathbf{28} + 8 + 20 + 16 + 12 + 24 + 4 + \mathbf{28} + 4 + 24 + 12 + 16 + 20 + 8 + \mathbf{28}$$

$$= 28 + 28 + 28 + 28 + 28 + 28 + 28 + 28 + 28$$

$$= 28 * 9$$

$$= 252.$$

Teaching Suggestions

Lillian's friend Molly actually thought up this pastime! After Lillian offered it as the basis for a problem the discussion at the Math Forum was just how much we should "give away"? Should there be graphics? If we included graphics, how many should be included? Should we just present the problem in text?

Just as your students work hard to solve our problems, we work hard to write good problems. We thought you might find it interesting to read some of the thoughts that were shared as we discussed this problem:

> *I wonder if the examples make it too clear that patterns repeat for multiples of the base window sizes - discovering that was what made the problem fun for me.*

> *I'm a little bothered by showing so clearly with the pictures that the path scales as the window scales. To me that is the key to solving the problem and discovering that was the fun part of doing it. What if you just showed the 2 by 3 and the 6 by 8 so they got to see examples of where it sometimes hits all and sometimes doesn't, but didn't make the scaling so obvious? Or would not giving the scaling hint tend to make kids just draw the 28 by 36 and figure it out by hand? For example, would it not occur to them to check for scaling?*

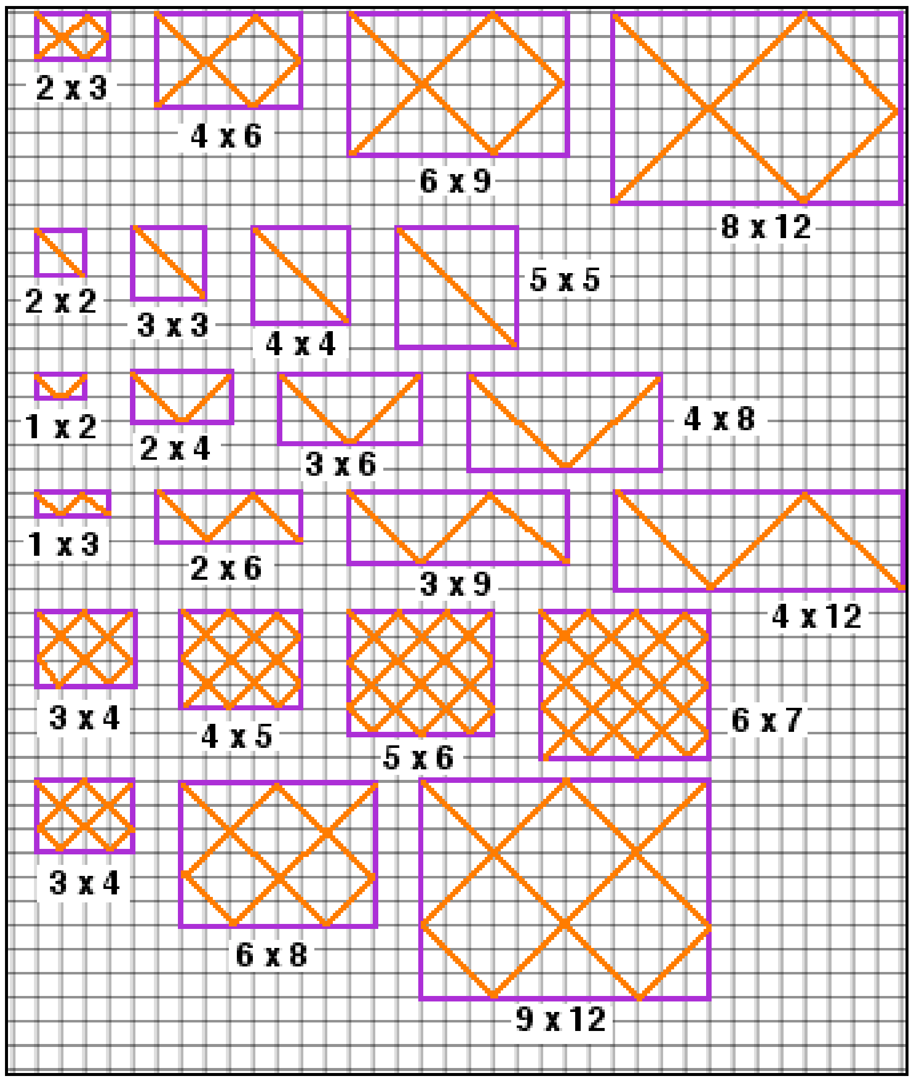
> *I actually would consider including more graphics rather than fewer. For my way of thinking, I needed to view a lot of pictures before getting a handle on what was happening.*

> *I guess I am tempted to agree that the fun of this problem comes from making discoveries, especially since there are ways besides thinking about scaling that are valid approaches. I like the idea of giving a number of pictures, only one pair of which is a multiple. Like maybe giving 2 x 3, 4 x 6, and 3 x 4? That way, the multiples are there, but they aren't the only focus of the pictures we give.*

It's interesting because on the face of it, this isn't a hard problem -- you could always just draw the picture and answer it. It might be a hard problem to see patterns in (though I'm not convinced -- middle-schoolers have a nice combination of patience with straightforward tasks, and interest in seeking patterns). One thought is that we go ahead and give all four pictures in the problem itself, and in the packet and activity series, give suggestions on making the problem more open-ended because it's those classrooms in which the students will have more problem-solving resources to draw on?

And by the way, while standing out in the back yard yesterday, I was looking at the old pottery shed which has multi-paned windows of various sizes, and I found myself trying to draw these paths in my head! And I discovered that I'm not very good at it - I had a very hard time following the reflections off the outer edge near the corners in my mind's eye. But I thought it was fun that I found myself trying it - I think that's a sign of an interesting problem.

When you present this problem to your students you might consider reading the text aloud to them and have them create some of the "pictures" as they make sense of the text. Some of those pictures drawn might be:



Resist the urge to give direct instructions on a specific approach. Ask students to paraphrase the problem to check on their understanding before they begin working on it. Ask questions that help them understand the language of the problem, visualize it, and discover patterns. Good questions help students clarify their thinking and give you useful information as well.

The questions in the Answer Check, above, might serve as good prompts to help students make progress. Encourage students to use a strategy that works for them. You can see from the various methods that we have thought to use for this problem that there are many ways to approach this problem. And, we may not have thought of them all!

Scoring Rubric

On the last page is the **problem-specific rubric,** to help in assessing student solutions. We consider each category separately when evaluating the students' work, thereby providing more focused information regarding the strengths and weaknesses in the work. A **generic student-friendly rubric** can be downloaded from the *Scoring Guide* link on any problem page. We encourage you to share it with your students to help them understand our criteria for good problem solving and communication.

We hope these packets are useful in helping you make the most of Pre-Algebra PoWs. Please let me know if you have ideas for making them more useful.

https://www.nctm.org/contact-us/

Pre-Algebra Scoring Rubric for *Lillian's Lines*

For each category, choose the level that *best describes* the student's work

	Novice	Apprentice	Practitioner	Expert
Problem Solving				
Interpretation	does none or one of the things listed under Practitioner	does two of the things listed under Practitioner	understands that the path always starts at the top left corner and keeps going until it "arrives" in another corner of the outer rectangle understands that the path is at a 45° angle to the edges of the rectangle understands that what is being counted is the number of squares, or panes, the path goes through	is at least a Practitioner in Strategy and has successfully answered the Extra
Strategy	does not have any ideas about how to solve the problem	has some ideas about how to solve the problem, but isn't quite there	has a strategy that relies on skill, not luck might draw the 28 x 36 solution and just count *or*, using a simpler problem, might relate the 28 x 36 problem to the 7 x 9 and work from there, *or* might find a rule or a pattern to help them predict the solution for the 28 x 36 case	uses two different methods
Accuracy	has made many errors	makes a few errors that lead to an incorrect answer	makes no arithmetic mistakes that really matter	[not normally available for this category]
Communication				
Completeness	has written nothing that tells you how they found their answer	shows work without an explanation *or* explains everything without showing the numbers doesn't include enough information for another student to follow	attempts to explain all of the steps taken to solve the problem, which might include: • what, if any, patterns they noticed • how the examples helped • other window sizes drawn and tested • how they knew their drawing was correct	adds in useful extensions and further explanation of some ideas involved, which might include: • what was hard about the problem, and how they made simpler versions to look for patterns • the least common multiple
Clarity	explanation is very difficult to read and follow	another student wouldn't be able to follow their explanation entirely long and written in one paragraph lots of spelling errors/typos	explains all of the steps mentioned in such a way that another student would understand makes an effort to check their formatting, spelling, and typing (a few errors are fine)	formats things exceptionally clearly answer is very readable and appealing
Reflection	*The items in the columns to the right are considered reflective, and could be in the solution or the comment they leave after viewing our answer:*	checks their answer (not the same as viewing our "answer check") reflects on the reasonableness of their answer	connects the problem to prior knowledge or experience explains where they're stuck summarizes the process they used	comments on and explains the ease or difficulty of the problem **revises their answer and improves anything**
	does nothing reflective	does one reflective thing	does two reflective things	does three or more reflective things or an great job with two