Pre-Algebra PoW Packet

Integer Images

Problem 3572 • https://www.nctm.org/pows/

Welcome

This packet contains a copy of the problem, the “answer check,” our solutions, some teaching suggestions, and samples of the student work we received in November 2005. The text of the problem is included below. A print-friendly version is available using the “Print” link on the problem page.

Standards

In ***Integer Images***, students are asked to use the four equations to figure out which integers the quadrilateral and the circle represent. The **key concepts** are variables, unknowns, variable expressions and “word” expressions/equations.

If your state has adopted the Common Core State Standards, this alignment might be helpful:

Grade 6: Expressions & Equations

Apply and extend previous understandings of arithmetic to algebraic expressions.

Reason about and solve one-variable equations and inequalities.

Grade 7: Expressions & Equations

Solve real-life and mathematical problems using numerical and algebraic expressions and equations.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively
3. Construct viable arguments and critique the reasoning of others.
6. Attend to precision
7. Look for and make use of structure
8. Look for and express regularity in repeated reasoning.

The Problem

Integer Images

This puzzle has two unknowns and four equations, or clues. I am trying to figure out what integer could be substituted for the circle and what integer could be substituted for the quadrilateral so that all four statements are true.

$$(1 \cdot \bigcirc)(2 \cdot \Diamond) = 48$$

$$\bigcirc - \Diamond = -5$$

$$\Diamond^2 + \bigcirc^2 = 73$$

$$\frac{3 \cdot \Diamond}{8 \cdot \bigcirc} = 1$$

What integers do the circle and the quadrilateral represent?

Extra: Which equation has the fewest integers that would work?

Answer Check

After students submit their solution, they can choose to "check" their work by looking at the answer that we provide. Along with the answer itself (which never explains how to actually **get** the answer) we provide hints and tips for those whose answer doesn't agree with ours, as well as for those whose answer does. You might use these as prompts in the classroom to help students who are stuck and also to encourage those who are correct to improve their explanation.

The circle represents 3. Be sure you also find the integer represented by the quadrilateral.

If your answer **doesn't** match ours,

- think about all of the equations before you start solving the problem.
- did you remember to consider negative numbers?
- did you remember to check your numbers in each of the four equations?
- did you check your arithmetic?

If any of those ideas help you, you might *revise* your answer, and then leave a comment that tells us what you did. If you're **still stuck**, leave a *comment* that tells us where you think you need help.

If your answer **does** match ours,

- did you try the Extra?
- have you clearly shown and explained the work you did?
- have you tried using formal algebra?
- are you confident that you could solve another problem like this successfully?
- did you make any mistakes along the way? If so, how did you find and fix them?
- are there any hints that you would give another student?

Revise your work if you have any ideas to add. Otherwise leave us a *comment* that tells us how you think you did—you might answer one or more of the questions above.

Our Solutions

Method 1: Start with Equation 1

We decided to look at each of the four equations in order. When we looked at the first equation we knew that we could apply the Associative and Commutative properties to simplify it to have:

2 * circle * diamond = 48

If we divide each side of that equation by 2, we have:

$$\frac{2 * circle * diamond}{2} = \frac{48}{2}$$
$$circle * diamond = 24$$

We made a list of the factor pairs of 24 thinking that those numbers would be the values of circle or diamond:

1, 24, 2, 12, 3, 8, 4, 6

The second equation says that when we subtract the two numbers the result has to be -5. If we subtract the 8 from the 3 we'll get a -5. We put 3 in for the circle and 8 for the diamond and 3 - 8 = -5, so that works.

We tried those for third equation to make sure they work.

$$3^2 + 8^2 = 73$$
$$9 + 64 = 73$$
$$73 = 73$$

And we tried them in the fourth equation:

$$\frac{3 \bullet 8}{8 \bullet 3} = 1$$
$$\frac{24}{24} = 1$$
$$1 = 1$$

They work in all four equations. The circle is 3 and the diamond is 8.

Method 2: Start with Equation 3

I looked at the four equations and the third one looked easiest to start with because it didn't seem like there would be that many integers that would work. I made a list of the square numbers under 73.

1, 4, 9, 16, 25, 36, 49, 64

I figured out that 9 and 64 were the only ones that would add to 73. So the two numbers could be 3 and 8. Now I have to figure out which one is the circle and which one is the diamond.

I decided to use the fourth equation. If the diamond is 3 and the circle is 8, then the left side is 9/64, and that's not 1. So the diamond must be 8 and the circle must be 3.

I checked those numbers in the other two equations. In the first one, I got 3 • 2 • 8, and that's 48, so that works. In the second one, I got 3 - 8, which is -5, and that works, too.

The circle is 3 and the diamond is 8.

Method 3: Start with Equation 4

I looked at the fourth equation and knew that the top and bottom had to be equal so that the whole thing was 1. At first I thought I could only put 8 on top and 3 on the bottom. But then I realized that I could use 16 and 6 or 24 and 9, or anything that reduces to $\frac{8}{3}$.

Then I looked at the third equation and I could see that 8 and 3 were the only ones that would work. All the rest of the combinations would be too big. So the numbers must be 8 and 3. I checked them in first equations:

$$(1 \bullet 3)(2 \bullet 8) = 48$$
$$3 \bullet 16 = 48$$
$$48 = 48$$

And then I checked them in the second equation:

$$3 - 8 = -5$$
$$-5 = -5$$

They worked in all four equations! The circle is 3 and the quadrilateral is 8.

Extra: I think that the third equation has the fewest integer solutions. There were a lot of answers for the fourth one. There are a lot for the second equation, since the two numbers just have to differ by 5. There are also a lot for the first one, since the two numbers have to multiply to 24 and there are a lot of whole numbers that would work. But for the third equation, you can only have 8 and 3 or you could have -8 and -3.

Method 4: Make a Table

We will use C for the circle and D for the diamond. The first equation is something that we can simplify:

$$(1 \bullet C)(2 \bullet D) = 48$$
$$1 \bullet C \bullet 2 \bullet D = 48$$
$$2 \bullet C \bullet D = 48$$
$$C \bullet D = 24$$

The product of the two answers must be 24. That leaves us with a lot of possibilities. Here is a list of the factor pairs of 24. Since order matters, we will list all of them. Ah, but wait! That's only half of the possible solutions. The factors could also be negative, too:

1	24
2	12
3	8
4	6
6	4
8	3
12	2
24	1

1	24		-1	-24
2	12		-2	-12
3	8		-3	-8
4	6		-4	-6
6	4		-6	-4
8	3		-8	-3
12	2		-12	-2
24	1		-24	-1

The second equation tells us that C - D = -5. The factor pairs that have a difference of 5 are the 3 and 8 and also the -8 and -3.

$3 - 8 = -5$ and $-8 - (-3) = -5$

The third equation says that the squares of the two numbers sum to 73. Let's try them:

$$8^2 + 3^2 = 64 + 9 = 73$$

or

$$(-3)^2 + (-8)^2 = 9 + 64 = 73$$

Those work fine. So here's where we are. There are two possibilities.

circle (C) = 8 and diamond (D) = 3
or
circle (C) = -3 and diamond (D) = -8

Let's try each of these in the last equation.

$$\frac{3 \bullet 8}{8 \bullet 3} = \frac{24}{24} = 1$$

$$\frac{3 \bullet -3}{8 \bullet -8} = \frac{-9}{-64} = ?? \text{ (not 1!)}$$

The only one that works is for the circle to be 3 and the diamond to be 8, so that must be the answer.

Method 5: Algebra (We would not expect Pre-Algebra students to use formal algebra.)

Let x represent the circle and y represent the quadrilateral. I can write the four equations as:

$(1x)(2y) = 48$ or $2xy = 48$ or $xy = 24$

$x - y = -5$

$y^2 + x^2 = 73$

$$\frac{3y}{8x} = 1$$

I'm going to work with the first two equations. If I know that $x - y = -5$ then I also know that $x = (-5 + y)$ and so I can write:

$$(-5 + y)(y) = 24$$
$$y^2 - 5y = 24$$
$$y^2 - 5y - 24 = 0$$
$$(y - 8)(y + 3) = 0$$

From this we know y = 8 or y = -3. If y is 8 then using xy = 24, x is 3. If y is -3, then x is -8. We try these values for x and y in the fourth equation to see which one(s) work.

$$\frac{3 \bullet 8}{8 \bullet 3} = \frac{24}{24} = 1$$

but

$$\frac{3 \bullet -3}{8 \bullet -8} = \frac{-9}{-64} \neq 1$$

Only (3, 8) work. Now I'll check the third equation:

$$8^2 + 3^2 = 64 + 9 = 73$$

It works. Now I'll check the second equation:

$(3) - (8) = -5$

It works. The circle is 3 and the quadrilateral is 8.

Teaching Suggestions

When we first offered this problem, quite a few people came up with the correct solution. The fourth equation was the source of the most common mistake we saw. A lot of people said that the only possible solutions were 8 and 3, so that you would get $\frac{24}{24}$. But in fact, there are infinitely many solutions! To start thinking about what all those answers might look like, here are a few we considered and we know there are plenty more:

$$\frac{3 \cdot 16}{8 \cdot 6} = \frac{48}{48} \qquad \frac{3 \cdot 24}{8 \cdot 9} = \frac{72}{72} \qquad \frac{3 \cdot 32}{8 \cdot 12} = \frac{96}{96}$$

Given the other equations, it's easy to see that the only solution to the fourth one that will work in the others is the 8 and 3. But that doesn't mean that there's only one possible solution for the fourth equation.

A few people said that the solutions were -8 and -3. It's great that they considered negative solutions (very few students did), and it's true that you could put those numbers in each equation and have it work. However, you can't put them in for the same shape (circle or square) in each one and have them work. For example, for the second equation to be true, we must have -8 - (-3) = -5, so the circle is -8. But now jump to the fourth equation. If the circle is -8, that's not going to work.

I was talking to a teacher about first presenting this problem to students using the Scenario, which features only the graphic display of the four equations. She wondered out loud whether her students would realize that if they found a number for the circle that worked in the first equation that they would necessarily have to use that same number in the second equation. My suggestion to her was to let her students work with the equations in whatever way they thought to approach them. Would it hurt to have them work with the equations independently? I don't think so because once they have more restrictive conditions they'll have a great starting point.

We named this problem *Integer Images* thinking that the word "integer" might be enough of a hint to consider negative as well as positive numbers. Maybe those students using Noticing and Wondering will have that realization!

The questions in the Answer Check, above, might serve as good prompts to help students make progress. Encourage students to use a strategy that works for them. You can see from the various methods that we have thought to use for this problem that there are several ways to approach this problem. And keep in mind that we may not have thought of them all!

Sample Student Solutions

focus on **Interpretation**

In the solutions below, I've provided the scores the students would have received in the **Interpretation** category of our scoring rubric. My comments focus on what I feel is the area in which they need the most improvement.

Novice	Apprentice	Practitioner	Expert
Understands few of the concepts listed in the Practitioner column.	Understands most but not all of the concepts listed in the Practitioner column.	Understands that • the problem asks what integer each shape represents. • the integers (numbers) they choose must work in all four equations. • whatever number is represented by the circle must represent the circle in all equations–order matters.	An Expert would consider negative numbers at every turn, as in the fourth method shown in the Teacher Packet.

Laini
age 10

Interpretation
Novice

The circles and the quadrilaterals represent the 0 and 5.
The thired one.

I looked for two numbers that could fit in with all of the equations, but there was one equation that those numbers did not go with an I think that one was number three.

I notice that Laini's first statement sounds as though she's on the right track because she is looking for two numbers that work with the four equations. I wonder, though, how she decided on 0 and 5 as her answers. As a starting point I would ask her to tell me more about how she found those numbers.

Jazmin
age 12

Interpretation
Novice

i don't know but i think 24 x 2

i don't know i was very hard for me but i think

At first glance it seems that Jazmin has no idea about the problem but since 24 times 2 equals 48 it seems that she was looking at the first equation.

I might ask her what two numbers multiplied equal 24. If she responds with one or more pairs of numbers next I would ask her if those two numbers work in one of the other equations.

Itamar
age 11

Interpretation
Apprentice

The answer is the circle equals 3 and the square equals 8.

1*x*2*y=48 and x-y=-5. Therefore, x, or circle, equals 3, and y, or sideways square, equals 8, because only those answers fit both equations.

Itamar's use of algebraic notation is a sophisticated approach and I might be convinced that he has interpreted this problem at a Practitioner level. When I refer to the rubric, however, I'm reminded that he should have mentioned all four equations and not just the two he references.

Besides asking him about the other two equations I'll ask him how he knew that "only those answers fit both equations." What did he do to decide that was true?

Shannon
age 13

Interpretation
Apprentice

My answers to these problems were first: 8,3 Second: -8,-3 Third: 3,8 And last was 8,3.

On the first step of this problem, I saw that 1*8=8 and 2*3=6 so 8*6=48. Then on the second step, -8-(-3)=-5. Then on the third step I saw that 3 to the second power=9 and that 8 to the second power=64 and 64+9=73. Then on the last step, I saw that 3*8=24 and 8*3=24 so 24/24=1.

I notice that Shannon found similar but different answers for each of the four equations.

I wonder if she saw that the problem stated that there were "two unknowns." I might direct her to that first sentence of the problem.

Molloy
age 12

Interpretation
Practitioner

The quadrilateral is eaqual to 8 and the circle is eaqual to 3. Extra: The equation that the fewest intergers would work in is the last one. That is why I used it to help me solve the equation, there is only one answer.

To solve the problem, first I had to see which equation had the least number of answers. That was the last one for the fraction to equal 1, the numerator and the demonimator had to be equal so 3*_= 8*_ The first blank has to be 8 and the second blank has to be 3. So both sides equal 24. 24 over 24 equals 1. So if the first blank is a quadrilateral which equals 8, and the second blank is a the circle, which is 3, then the quadrilaterals for the other quadrilaterals and circles are equal to the ones in this equation.

So I tried it in the first equation.

c=circle q=quadrilateral
(1*c)(2*q)=48
(1*3)(2*8)=48 TRUE

Then I tried the second one
c-q=-5
3-8=-5 TRUE

Then I tried the third
$q^2 + c^2 = 73$
$8^2 + 3^2 = 73$ TRUE

I already did the fourth one, so I am 100% sure that the circle is 3 and the quadrilateral is 8.

Extra: The last one is the equation with the fewest intergers because it only has one possible answer.

Molloy has done a nice job showing how he worked from equation four and then tested each of the other equations to see if his answers worked.

I would score his as a Practitioner in Interpretation but as an Apprentice in Accuracy because of his statements about the fourth equation having only one solution. As explained in the Teaching Suggestion section of this Packet that's not an accurate statement.

I might give him one of those other possible values for the circle and quadrilateral and ask him what he thinks.

Reflecting on that idea might also have him change his mind about his response for the Extra.

Nicole
age 15

Interpretation
Practitioner

the circle = 3 the diamond = 8

First, I divided 48 by 2 because I multiplied 1 and 2. So, to get all of the integers on one side, I divided 2 from both sides and saw that the product of the diamond and the circle had to equal 24. For the second problem, I took the possiblilities for both numbers and tried to plug them in. I found that 3 and 8 worked.

To double check, I tried to square both of them to see if they would equal 73, which they did. And for the last problem, once the numbers qere multiplied, I got the answer of : 24/24 which is also equal to 1. Extra: problem 2.

(1*3) (2*8) = 48√
3-8= -5√
$3^2+8^2=73$√
3*8 / 8*3=1√

Nicole could be encouraged to add more to her explanation so that another student could follow along better but that just means I would give her a score of Apprentice in Completeness. She has shown Practitioner level work for Interpretation.

Juli
age 12

Interpretation
Expert

the circle and the quadrilateral represent -8 and -3, the equation with the fewest intergers that work is the equation: -8squared + - 3squared= 73.

I guessed and check for each problem and once I found a resonable answer for the first one, I would check to see if it worked for the second, third...etc. problems. If it did I would than see if it worked for all four. That's how I came to get -8 and -3 throght quess and check. For the equation that has the fewest intergers that would work, I thought the first one wouldnt work because each number has a lllllllllloooooottt of possibilities because 1x? x 2x? to get a fairly common answer if pretty easy. there are also many posibilities to use any two numbers to get -5. And for the last equation I knew it wasnt the fewest interger problem because alot of numbers, if divided together, can get 1. Thats how I knew that the third equation was correct.

I notice that Juli considered negative numbers and, while I would like more details about what she guessed and checked to get -8 and -3, I would score her as an Apprentice in Completeness rather than lowering her score for Interpretation.

I would also score her as an Apprentice in Accuracy since her two integers do not satisfy the second equation. If I ask her to show how they work in the second equation she may realize that she overlooked the order in which the circle and the quadrilateral occur.

Scoring Rubric

A **problem-specific rubric** can be found linked from the problem to help in assessing student solutions. We consider each category separately when evaluating the students' work, thereby providing more focused information regarding the strengths and weaknesses in the work.

We hope these packets are useful in helping you make the most of Pre-Algebra Problems of the Week. Please let me know if you have ideas for making them more useful.

https://www.nctm.org/contact-us/