NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Problem of the Week Teacher Packet

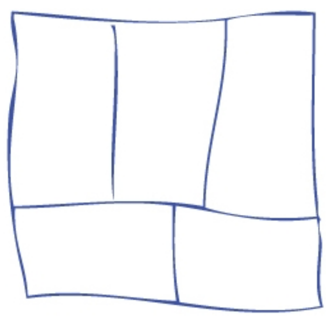
Arranging Rectangles

Give the coordinates of the vertices of a triangle that's similar to the one shown and which has a perimeter three times that of the given triangle.

Kasim had the following challenge for homework:

Is it possible to arrange five congruent rectangles without gaps or overlaps so that they form a square if the rectangles cannot all be oriented the same way?

He drew this picture:



1. Help him decide whether that arrangement of three "vertical" rectangles on top and two "horizontal" rectangles on the bottom could ever be a square.
2. Come up with at least three other ways to arrange the rectangles and show whether or not each arrangement could be a square.

Answer Check

After students submit their solution, they can choose to "check" their work by looking at the answer that we provide. Along with the answer itself (which never explains how to actually get the answer) we provide hints and tips for those whose answer doesn't agree with ours, as well as for those whose answer does. You might use these as prompts in the classroom to help students who are stuck and also to encourage those who are correct to improve their explanation.

> *His arrangement of rectangles can never form a square. In fact, no arrangement of five congruent rectangles, that are not oriented the same way, without gaps or overlaps can ever form a square.*

If your answer does not match our answer,

- did you make sure that all five rectangles are not facing the same direction?
- what happens if you make the short side of the rectangle 1 unit long? How do all of the other dimensions work out?
- how many short sides of the rectangle does it take to make one long side? Is it the same no matter which side of the square you use for comparison?

If these ideas helps you, you might revise your answer, and then leave a comment that tells us what you did. If you're still stuck, leave a comment that tells us where you think you need help.

If your answer does match ours,

- did you solve it for a specific size rectangle? Be sure your conclusion works for any size rectangle. (You might use a variable for the length of the short side of the rectangle.)

- are there any hints you would give another student?
- did you make any mistakes along the way? How did you find them?
- what hints would you give another student?

Revise your work if you have any ideas to add. Otherwise leave us a comment that tells us how you think you did—you might answer one or more of the questions above.

Our Solutions

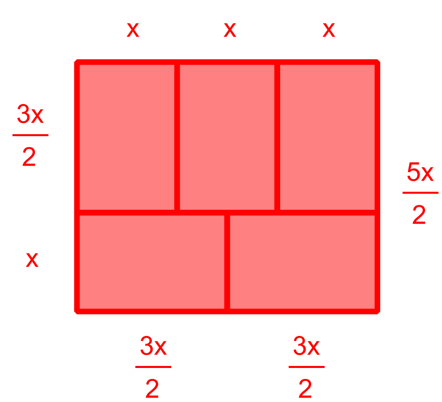
Question 1

We are told that one growth ring is about 1/8" thick. Looking at the picture, we can see that the rings are concentric circles, around the center of the tree. So we want to find the radius of the tree. Then we know that every inch of radius represents about 8 years of growth.

Looking at Kasim's arrangement of the rectangles, at first I wasn't sure what to do, since we don't know any of the lengths. But then I thought about the fact that if the resulting figure is supposed to be a square, that means all four sides will be the same length. Since I don't know any of the lengths, I decided to assign variables to the length and width of the small rectangles and then see what happens if I write some equations based on those variables and the fact that all four sides of the square have to be the same length.

We can assign a variable to the short side of the small rectangle. Call the short side x. Then the top of the large rectangle has a length of 3x. This means the bottom must also have a length of 3x (since the opposite side of a rectangle must be the same length).

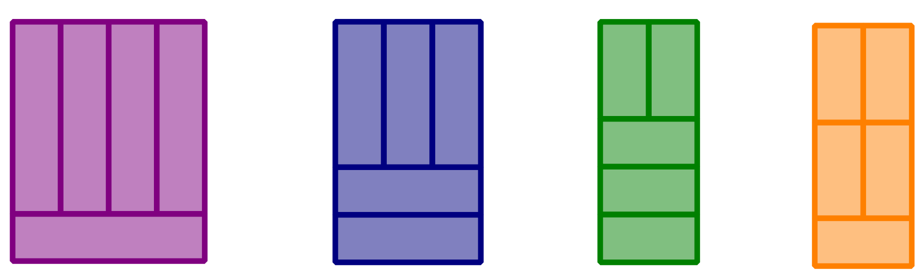
Since two long sides of the small rectangle form the bottom of the large rectangle, each must have a length of 3x/2. The left and right sides of the large rectangle are made of a short side of the small rectangle and a long side. This means that in terms of x, they have a length of x + 3x/2. That totals 5x/2.



This means that the large rectangle can't be a square – the top and bottom have a length of 3x and the left and right have a length of 5x/2. Since they're not equal, it's not a square.

Question 2

There are at least six other arrangements of five congruent rectangles (we found it hard to prove that there weren't any others). Four of them are pictured below. The middle two could be altered so that the horizontal rectangles were split up (one on the top and one or two on the bottom).



It can be shown that none of these could possibly be squares, using the same technique used in Question 1. Namely, assign a variable, such as x, to the short side of the small rectangle and calculate the lengths of the remaining sides of the large rectangle. In no case will the resulting figure be a square. (The small rectangles in the pictures are all congruent, and have the only possible ratios of length to width that will result in the large figure being a rectangle.)

Standards

The properties of squares and rectangles are covered in the elementary grade standards. Using variables to solve a problem like this for the general case, using those properties, is most closely suited to the High School Algebra standards, though it doesn't align to any one standards so neatly. This problem, particularly for geometry students, is really a chance to focus on the Mathematical Practices, especially #1.

If your state has adopted the Common Core State Standards, you might find the following alignments helpful. Similarity isn't a high school standard, but it's possible the context will provide some challenge.

High School: Geometry: Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.

Teaching Suggestions

There are a number of possible ways students will tackle this problem. A common one might be to try guessing dimensions for the small rectangle and seeing if the resulting large rectangle is a square. That's a great way to start! In fact, that's an excellent problem solving strategy, especially when you're stuck. Stick some numbers in and see what happens. Often times this will lead you to relationships and patterns that will lead to a more general solution. In this case, it may lead students to notice that they only need to assign a number to the short side of the small rectangle and everything else will fall out from that. Then they're just one (sometimes big!) step away from trying a variable for one of the dimensions and seeing what happens.

Some students may try a bunch of different dimensions and decide that since they didn't find a way to make it a square, it can't ever be a square. Encourage them to think about how they could solve it for the "general" case, when we do not rely on a specific length or quantity. This is where the utility of algebra becomes apparent.

Coming up with alternative arrangements may prove difficult for some students. They might benefit from using physical rectangles. While it's unlikely that the rectangles will have dimensions that will allow five of them to form a rectangle, they should help students discover some different possible arrangements.

For students who are ready for a challenge, you might just give them the Scenario Only. It includes a lot less scaffolding and leaves things considerably more open. It also provides an opportunity to talk about how to find all of the possible arrangements. We found this to be challenging in the office, and did not feel it was appropriate for all students. But we felt it could lead to some interesting conversations and explorations.

Sample Student Solutions – Focus on Strategy

In the solutions below, we've focused on students' "strategy", particularly around the first question. Generally speaking, this reflects whether the student has used a mathematically sound method to solve the problem that doesn't rely on luck. My comments focus on what I feel is the area in which they need the most improvement.

Novice	Apprentice	Practitioner	Expert
Has no ideas that will lead them toward a successful solution. Seems to rely solely on intuition.	Uses a strategy that uses luck instead of skill, or doesn't provide enough detail to determine. Might use specific numbers to show that the arrangement(s) can't form a square.	Uses a strategy that relies on skill, not luck, which might include using variables (including words) to show a logical inconsistency in each arrangement.	An Expert in Strategy might explore ways to show whether this will ever work (with a number of rectangles other than five).

Nick, age 12, Novice

The 2 small squares can be a square but the three can not.

If you stack the two on top of each other you get a square but the three bigger ones can not make a square in any combination.

Nick hasn't said anything about how he decided his statement was true (did he draw pictures? assign lengths?), but it seems like he doesn't have any ideas that will lead to a sound strategy without a lot of work. I am tempted to ask him how he decided the three rectangles can't make a square, but probably ought to point out that the five rectangles have to be arranged the way they are in Kasim's picture, and if he has any thoughts about whether it could be a square that way.

Dennis, age 10, Novice

The rectangles could never become a square because every side of a square is congruent and those sides are wobbly so even trying to make 3 different arrangements you can not make a square.

I got my answer by looking at the so called "SQUARE" and seeing the sides are not congruent so therefore no square can be composed with wobbley rectangles.

Dennis seems to be judging the given picture and not thinking more flexibly about the situation. I would agree with him that the given picture isn't a square, but would ask him if he could use a different size for the rectangles in Kasim's picture so that it is a square.

Larissa, age 13, Apprentice

1.The arrangement is possible for the rectangles to be a square. 2. There can be a square made from the diagram rotated 90, 180 and 270 degress.

1. For all of the heights(of the vertical rectangles) i will say they are 6 and for all of the widths(of the vertical rectangles) i will say they are 3. So the width of the the square will be 9(I added all of the widths of the vertical rectangles.) The horizontal rectangles will have widths of 6 and heights of 3. When i add the widths of the horizontal rectangles it equals 9. Then when i add the height of one vertical rectangle and one height of a horizontal triangle it equals 9. All of the sides equal 9 so the diagram is able to be a square.
2. If you rotate the diagram 90 degress it is a different arrangement but all the saides can still be the same. This goes for it rotating 180 and 270 degress.

Larissa has used a good initial strategy – try some numbers and see what happens. But she's said that all the sides of the square are 9, when in fact the bottom of the rectangle, made of two "heights", which are 6. I would point that out to her and ask her how that changes her explanation. Then I might ask her what happens if she tries some other numbers.

Kimberly, age 13, Apprentice

In order for the five rectangles to form a square, their length to width ratio must be 3:2. Other than that, the rectangles can never make a square if they are facing different ways.

I started off by drawing the five rectangles as they are shown on the actual problem (3 vertical and 2 horizontal) and I substituted in different lengths and widths for the rectangles. I came up with the ratio 3:2 for length to width as the only possible way for that certain arrangement to ever make a square.

If they were all to go a different way, not perfectly horizontal or perfectly vertical, then they could never make a square without overlapping in some way because a square must have 90° angles, according to the definition of a square.

It really depends on how you interpret "if the rectangles cannot all be oriented the same way". At first, I thought it meant that the rectangles could go vertical and horizontal, but as I started thinking about the different possibilities, I figured that they could really go in any direction you want them to.

I'm not entirely sure how Kimberly has reached her conclusion, so I'll ask her to tell me more about it by showing me more of the work that she did.

Andy, age 12, Apprentice

o. There is no way a grouping of five congruent rectangles without overlaps can ever form a square.

The rectangle that Kasim cannot be made to be a square, no matter what dimensions you put on each of the rectangles.

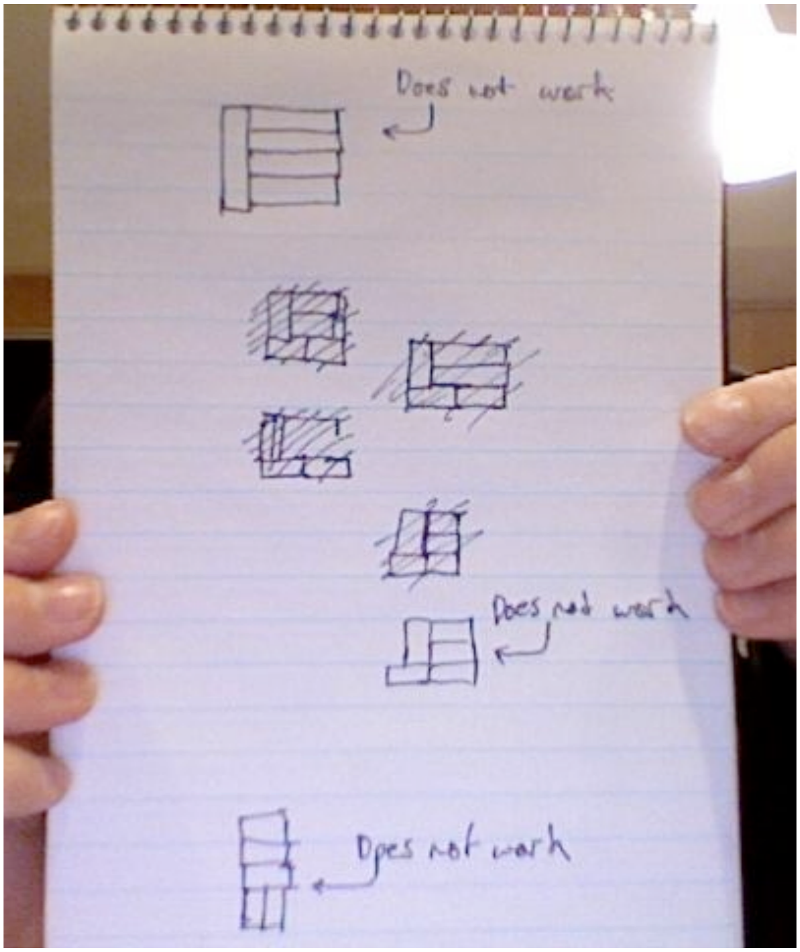
a. In order for the rectangle to be a square, three widths (w) have to equal 2 lengths (l)
Say that: w = 1 and l = 2

Using "three widths (w) have to equal 2 lengths (l)" this means we can set up this equation: 3w = 2l
If w = 1 and l = 2, then the equation can be simplified into: 3 = 4 which is not true.

b. If we just stick to the equation, then by dividing by 2, (3w)/2 = (2l)/2 and then 1.5 w = l
And by looking at his picture, 3w = l + w
If we substitute "1.5 w = l" into this equation, we get: 3w = 2.5 w
Which unless "w" is zero, is not true. ("w" cannot be zero, because then it would be a line)

Andy has a good idea to use variables to represent the dimensions of the small rectangles. But then he picks numbers to substitute for the variables instead of continuing to work with the variables to see if he reaches a conclusion without assigning values. I might ask him what happens if we say w = 2 and l = 3.

Looking at the picture, I came up with those four figures. each of these would not work.



The one on top:
$4w = l$ and "l" equals on side and "**$w + l$**" equals one side. Therefore: **$l = w + l$** and this is not possible unless **$w<0$** which is impossible.

The one in the middle:
Using the diagram, **$l = 2w$** and one side is "**$2l$**" and another is "**$l+w$**".
Therefore: **$2l = l + w$** and if **$l = 2w$**, then **$4w = 3w$** which is false

For the last one on bottom:
"**$3w + l$**" is one side length and "**$2w$**" is another. Therefore: **$2w = 3w + l$** which is clearly false.

This problem wasn't as hard as the others, but I did have some trouble finding other patterns for the 5 congruent rectangles. This was the reasons for the shapes that aren't squares. I appreciate you sending this, and it was nice to have an easy problem once in a while. Thanks again, and I really appreciate you taking the time to post this.

Roarke, age 13, Apprentice

No!It does not work. The sides of the polygon are not all equal.

I made a square with 6 inch sides. The long side of the rectangle is 3 inches. The short side is 2 inches. The top and the bottom of the polygon turn out to be 6 inches, but the sides only turn out to be 5 inches - so it is not a square.

Roarke has explored one possible set of dimensions for the rectangles. I might ask him if he tried any others and what happens when he does.

Part 2

You can set up your rectangles with three in a row on top lengthwise, with two on the bottom, and it doesn't make a square.

You can set up two on top with with the long side verticle, and the three on the bottom lengthwise, and it doesn't make a square.

You can set up you rectangles with the top row a lengthwise rectangle, the second and third rows have two verticle rectangles in them. It does not make a square.

Sara, age 14, Practitioner

Kasim's arrangement cannot be a square. Of the three configurations I came up with, none could be squares.

Sara has effectively used variables to represents the dimensions of the rectangles and shown how that leads to an inconsistency. I might ask her if she thought about whether there were other possible arrangements for the five rectangles.

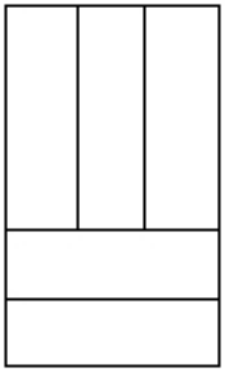
I began this problem by examining the diagram. The bottom side consists of two long sides of a rectangle. The top side is three short sides of a rectangle. And both the right and left edges are a long side and a short side. For this to be a square two long sides would have to equal three short sides. This is possible if there is a 2/3 ratio between a single short side and a single long side. So, if the long sides are 3 and the short sides are 2, then they are equal (3 x 2 = 2 x 3). However, two long sides *can never equal* one short and one long side, namely the edges. So, this cannot be a square, because the top and bottom can never be equal to the sides.

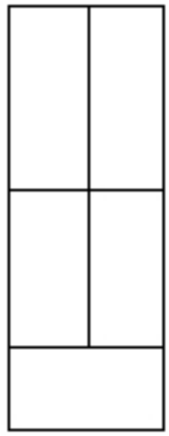
For my first five-rectangle combination, refer to the top left diagram. It is possible for the top and bottom to be equal, with a 1/3 ratio for short to long sides. In other words, if the short sides are 2, then the long sides are 6, and the top and bottom are equal. But this configuration can never be a square, because three short sides (top) would have to be equal to two short and one long side (right and left sides). This is impossible.

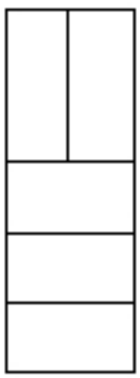
My second configuration is the top right rectangle. The top and bottom can be equal with a ratio of 1/2 for the short to long sides. If the short sides were 2, the long sides would be 4, and the top and bottom would be equal. However, two long and one short side (right and left sides) would have to be equal to a single long side (bottom), which is both impossible and ridiculous. This shape can't be a square.

My final shape is on the bottom left. Again, a 1/2 ratio would make the top and bottom sides equal. If the short sides are 4, then the long sides are 8, and the top and bottom are equal. But two short sides (top) would have to equal three short and one long side (right and left sides) for this shape to be a square. Because this is impossible, none of the shape can possibly be squares.

Note: Please pretend the rectangles in each individual diagram are congruent to the other rectangles in that diagram. Thank you.







Scoring Rubric

A **problem-specific rubric** can be found linked from the problem to help in assessing student solutions. We consider each category separately when evaluating the students' work, thereby providing more focused information regarding the strengths and weaknesses in the work.

We hope these packets are useful in helping you make the most of Geometry Problems of the Week. Please let me know if you have ideas for making them more useful.

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